KUWAIT UNIVERSITY

Department of Mathematics & Computer Science

Math 102 (Calculus B) — Second Examination — July 19, 2008

Duration: 90 minutes

Calculators, Cell phones and Pagers ARE NOT ALLOWED.

Answer all of the following questions.

1. Evaluate the following integrals. (3 points each)

(a) $\int \cos(\ln x)dx.$

(b) $\int \frac{\tan^3 x \sec^4 x + \sin x}{\sec^3 x}dx.$

(c) $\int \frac{x}{\sqrt{8+2x^2-x^4}}dx.$

(d) $\int \frac{7x^2+x+3}{(x+1)(2x^2+1)}dx.$

(e) $\int \frac{\sec x}{2\sec x+\tan x+1}dx.$

(f) $\int \frac{dx}{x\sqrt{\sqrt{x}-4}}.$

(g) $\int (1+x)e^x\sqrt{1-(xe^x)^2}dx.$

2. Determine wether the integral $\int_0^1 \frac{dx}{e^x-1}$ is convergent or divergent. Evaluate it, if convergent.

(4 points)

Solution Key

1. (a) Using integration by parts twice

$$\begin{aligned}\int \cos(\ln x)dx &= \int (x)'\cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx = x\cos(\ln x) + \int (x)'\sin(\ln x)\\ &= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)dx\\ \int \cos(\ln x)dx &= \frac{x}{2}\left(\cos(\ln x) + \sin(\ln x)\right) + C\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{\tan^3 \sec^4 x + \sin x}{\sec^3 x}dx &= \int \left(\tan^3 x \sec x + \sin x \cos^3 x\right) dx\\ &= \int \left(\sec^2 x - 1\right) d(\sec x) + \int \left(\sin x - \sin^3 x\right) d(\sin x)\\ &= \frac{\sec^3 x}{3} - \sec x + \frac{\sin^2 x}{2} - \frac{\sin^4 x}{4} + C\end{aligned}$$

(c) Using the substitution $u = x^2$

$$\begin{aligned}\int \frac{x}{\sqrt{8 + 2x^2 - x^4}}dx &= \frac{1}{2}\int \frac{1}{\sqrt{8 + 2u - u^2}}du = \frac{1}{2}\int \frac{1}{\sqrt{9 - (u-1)^2}}du\\ &= \frac{1}{2}\sin\frac{u-1}{3} + C = \frac{1}{2}\sin\frac{x^2 - 1}{3} + C\end{aligned}$$

(d) Partial Fractions: $\frac{7x^2+x+3}{(x+1)(2x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{2x^2+1}$ $\quad (A = 3,\ B = 1,\ C = 0)$

$$\int \frac{7x^2 + x + 3}{(x+1)(2x^2+1)}dx = \int \frac{3}{x+1}dx + \int \frac{x}{2x^2+1}dx = 3\ln|x+1| + \frac{1}{4}\ln(2x^2+1) + C$$

(e) Using the substitution $u = \tan\frac{x}{2}$, $\sin x = \frac{2u}{1+u^2}$, $\cos x = \frac{1-u^2}{1+u^2}$, $dx = \frac{2}{1+u^2}du$

$$\begin{aligned}\int \frac{\sec x}{2\sec x + \tan x + 1}dx &= \int \frac{1}{2 + \sin x + \cos x}dx = \int \frac{1}{2 + \frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}}\frac{2}{1+u^2}du\\ &= \int \frac{2}{2(1+u^2) + 2u + 1 - u^2}du = \int \frac{2}{u^2 + 2u + 3}du\\ &= \int \frac{2}{(u+1)^2 + 2}du = \frac{2}{\sqrt{2}}\tan^{-1}\frac{u+1}{\sqrt{2}} + C\\ &= \sqrt{2}\tan^{-1}\frac{1}{\sqrt{2}}\left(\tan\frac{x}{2} + 1\right) + C\end{aligned}$$

(f) Using the substitution $x = u^4$

$$\int \frac{dx}{x\sqrt{\sqrt{x} - 4}} = \int \frac{4u^3}{u^4\sqrt{u^2 - 4}}du = 2\sec^{-1}\frac{u}{2} + C = 2\sec^{-1}\frac{\sqrt[4]{x}}{2} + C$$

(g) Using the substitution $xe^x = \sin\theta$, $(xe^x)' = (1+x)e^x$

$$\begin{aligned}\int (1+x)e^x\sqrt{1-(xe^x)^2}dx &= \int \sqrt{1-(xe^x)^2}d\,(xe^x) = \int \cos^2\theta d\theta \\ &= \frac{1}{2}\int (1+\cos 2\theta)d\theta = \frac{1}{2}(\theta + \frac{\sin 2\theta}{2}) + C \\ &= \frac{\sin^{-1}(xe^x)}{2} + \frac{\sin\left(2\sin^{-1}(xe^x)\right)}{4} + C\end{aligned}$$

2. The integral is divergent since

$$\int_0^1 \frac{dx}{e^x-1} = \lim_{c\to 0^+}\int_c^1 \frac{dx}{e^x-1} = -\lim_{c\to 0^+}\left[\ln(1-e^{-1}) - \ln(1-e^{-c})\right] = \infty$$